

# Mean velocity equation for turbulent flow: I. Application of Zwanzig-Mori technique

Jürgen Piest

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## Abstract

The hydrodynamic equation derived by N-particle statistical mechanics is investigated. The reason is an attempt to provide additional information concerning the closure problem of turbulence theory. In this paper, the third-order term of the friction force of the equation is calculated in terms of third- and fourth-order equilibrium time correlation functions.

## 1 Introduction

This is the first of a series of papers which deal with the hydrodynamic equation derived by N-particle statistical mechanics; especially with the calculation of the third order term of the friction force (divergence of the stress tensor).

The motive for the investigation is the closure problem of classical turbulence theory. It is well known that the problem arises when one declares, in case of turbulent fluid motion, the hydrodynamic velocity  $\mathbf{u}$  to be a random variable, and attempts to obtain an equation for its expectation by just building the expectation over the Navier-Stokes equation. The non-linear term then gives rise to an additional variable; when one formulates an additional equation for this, another variable is obtained, and so on. Apparently, if the basic equation of motion is non-linear, this method does not work. Instead, from the point of probability theory, it would be necessary to find the probability density  $f(\mathbf{u}, t)$  of the turbulent process, and to calculate the expectation as:

$$\langle \mathbf{u} \rangle = \int \mathbf{u} f(\mathbf{u}, t) d\mathbf{u} \quad (1.1)$$

$f$  is a functionally dependent on  $\mathbf{u}$ ; thus, (1.1) is a functional integral. As early as 1952, Hopf has formulated this theoretical approach [1]; a short account can be found in [2], ch. 4. - It became apparent that it is extremely difficult to provide an equation for  $\langle \mathbf{u} \rangle$  by following this approach. On the other hand, the success of direct numerical simulation (see, e. g. [3]) leads to the conclusion that it must be possible to calculate the mean velocity on the basis of the Navier-Stokes equation.

In this situation, it is perhaps reasonable to attempt the derivation of the mean velocity equation by N-particle statistical mechanics, in order to provide further information on the nature of the problem. It is obvious that for investigation of turbulent processes it is not necessary to deal with the molecular nature of the fluid. It is the probabilistic structure of statistical mechanics which constitutes the difference to Navier-Stokes theory. Statistical mechanics is the basis for describing a large set of physical processes including fluid dynamics; it would be a surprise if turbulent motion would be an exception. The complication is that in a statistical mechanics analysis it is necessary to distinguish between macroscopic and microscopic parts of the motion, and to handle the latter in a suitable way. - The Navier-Stokes equation can be derived from statistical mechanics via the Boltzmann equation by Chapman-Enskog method; see e. g. [4]. But in the course of the derivation of the Boltzmann equation the molecular chaos assumption is introduced, which is correct for many applications including laminar fluid motion, but seems problematic for a turbulence investigation.

An essential step is defining the mean velocity in the frame of statistical mechanics. The fluid velocity is defined:

$$\mathbf{u} = \frac{1}{\rho} \langle \mathbf{p} \rangle \quad (1.2)$$

$\rho$  is the mass density,  $\mathbf{p}$  the microscopic momentum density. In this paper, we consider incompressible constant density and temperature processes. Then  $\mathbf{u}$  is, up to a constant factor, equal to the expectation of  $\mathbf{p}$ . Under rather general conditions, the expectation is equal to the arithmetic mean of a time series of the microscopic quantity. Since for laminar flow fluctuations are microscopic,  $\mathbf{u}$  then is the "point" velocity of the flow as we usually understand it. On the other hand, in turbulent flow there appear macroscopic fluctuations; the arithmetic mean by definition averages over these also; thus, in this case  $\mathbf{u}$  is already the mean velocity. We have the peculiar situation that the statistical mechanics  $\mathbf{u}$  is equal the classical hydrodynamic  $\mathbf{u}$  or  $\langle \mathbf{u} \rangle$ , depending on whether the flow is laminar or turbulent. Therefore, the hydrodynamic equation in the form derived by N-particle statistical mechanics will, by way of trial, be considered the mean velocity equation.

The derivation by means of Zwanzig-Mori projection operator technique of statistical mechanics (POT) is utilized, in the presentation of Grabert [5]. For the convenience of the reader, some introductory material from an earlier paper of the author [6] is repeated. The hydrodynamic equation obtained by this technique shows a formula for the friction force containing a local equilibrium time correlation, which is a non-linear functional of the velocity; presently, there exists no theory from which it could be calculated. Therefore, it is necessary to develop it into a functional power series in the velocity; this presently restricts the applicability of the equation to low Reynolds number flow, just outside the Navier-Stokes regime. The coefficients of the series now contain total equilibrium correlation functions, which have been calculated by employing a separate theory.

In this paper, calculation of the third order term of the power series is described. Applying correlation theory is left to a consecutive paper. Performing a test with empirical data is still under way and will be reported later.

## 2 Hydrodynamic equation

This section is a somewhat altered version of section 2 of [6]. There, the description was restricted to stationary processes. In this paper, I provide the usual time-dependent formulas. The fluid is considered to be a system of  $N$  particles of mass  $m$  with positions  $\mathbf{y}_j$  and velocities  $\mathbf{v}_j$  which are combined to the phase space matrix  $z$ . Vector components are described by Latin indices, e. g.  $\mathbf{y}_j = \{y_{ja}\}$ . The system is enclosed in a box of Volume  $V$ . Later on, the thermodynamic limit is performed. - A function  $g(z)$  is called a phase space function, or microscopic variable. Especially, we need the space densities of the conserved quantities mass, energy and momentum  $n, e, \mathbf{p}$  which are collected to a 5-element linear matrix  $a$ . They are functions of an additional space variable  $\mathbf{x}$ :

$$a = \sum_{j=1}^N \tilde{a}_j \delta(\mathbf{x} - \mathbf{y}_j) \quad (2.1)$$

For the particle functions  $\tilde{a}_j$  we have  $\tilde{n}_j = m$ ,  $\tilde{\mathbf{p}}_j = m\mathbf{v}_j$ , while the energy function contains the interparticle potential.  $m$  ist the particle mass. Quantities (like  $a$ ) which are lists of 5 elements are denoted by normal letters while three-component vectors and tensors are bold. - The quantities  $a$  obey the conservation relations:

$$\dot{a} = -\nabla \cdot \mathbf{s} \quad (2.2)$$

The fluxes  $\mathbf{s}$  have the same general structure as the  $a$  (2.1); especially, we have  $\mathbf{s}_1 = \mathbf{p}$ . The time evolution of any phase space function  $A$  is described by the Liouville equation:

$$\dot{A} = \mathcal{L}A \quad (2.3)$$

$\mathcal{L}$  (often defined as  $i\mathcal{L}$ ) is the Liouville operator. From (2.3), the formal solution for  $A(t)$  given the initial value  $A$  is:

$$A(t) = e^{\mathcal{L}t} A \quad (2.4)$$

In the statistical model,  $z$  and  $N$  are considered random variables; that is, the probability density  $f(z, N)$  is of grand canonical type. The ensemble mean value (expectation) of a phase space function  $A$  is defined in the 'Heisenberg' picture:

$$\langle A \rangle(t) = \sum_{N=1}^{\infty} \int dz A(z, N, t) f(z, N) \quad (2.5)$$

In this formula,  $f(z, N)$  is the initial probability distribution, and  $A(z, N, t)$  is the value of  $A$  at time  $t$  if the initial positions and velocities of the particles are described by  $z$ . The operation (integration + Summation) is sometimes indicated by the symbol 'tr':

$$\text{tr}\{\Omega\} = \sum_{N=1}^{\infty} \int dz \Omega(z, N) \quad (2.6)$$

Certain probability densities (also called distributions here) are frequently used in the analysis. One of them is the (total) equilibrium distribution which corresponds to macroscopic rest:

$$f_0 = \psi(N) \exp(\Phi_0 + \beta(\mu N - H(z))) \quad (2.7a)$$

$$\psi(N) = \frac{1}{N!} \left(\frac{m}{h}\right)^{3N} \quad (2.7b)$$

Here,  $h$  is Planck's constant,  $\beta = 1/(k_B T)$ ,  $k_B$  being Boltzmann's constant and  $T$  the temperature,  $\mu$  is the chemical potential which is a function of mass density  $\rho = \langle n \rangle$  and temperature, and  $H(z)$  is Hamilton's function which describes the total energy of the fluid. For the normalization constant, we have  $\Phi_0 = -\beta P V$ ,  $P$  being the equilibrium pressure. Expectations with respect to the equilibrium distribution are denoted by  $\langle \rangle_0$ . In case of a simple fluid, the 'relevant probability distribution' of Grabert's formalism (see [5], sec. 2.2) is the local equilibrium distribution:

$$f_L(t) = \psi(N) \exp(\Phi(t) - a(z) \circ b(t)), \quad (2.8a)$$

$$b = \left\{ \beta \left( \frac{1}{2} u^2 - \frac{\mu}{m} \right), \beta, -\beta \mathbf{u} \right\}, \quad (2.8b)$$

$$\Phi(t) = -\log(\text{tr}\{\psi \exp(-a \circ b(t))\}). \quad (2.8c)$$

Here the symbol  $\circ$  is introduced for the operation: Multiplication, plus Summation over the 5 elements of the linear matrices  $a$ ,  $b$ , plus Integration over geometrical space. The elements of  $b$  are called the conjugate parameters; they are functions of the quantities  $\beta$ ,  $\mu$  and  $\mathbf{u}$  which we will sometimes call the thermodynamic parameters, and which will be considered to be slowly varying functions of space and time. The  $b$  are defined such that the expectations of the  $a$  are identical to their expectations in local equilibrium:

$$\langle a \rangle = \langle a \rangle_L \quad (2.9)$$

The POT is a means for separating macroscopic and microscopic parts of the random variables. It starts by defining the set of phase space functions which are relevant for the description of the process. For simple fluids, this set is identified with the densities of conserved variables,  $a$ . A projection operator is defined which projects out of any microscopic variable  $A$  the part which is proportional to the relevant variables. It reads:

$$\mathcal{P}A = \langle A \rangle_L + \langle A \delta a \rangle_L \circ \langle \delta a \delta a \rangle_L^{-1} \circ \delta a \quad (2.10)$$

Here,  $\delta a = a - \langle a \rangle_L$ ;  $\langle \rangle_L^{-1}$  denotes the inverse of the expectation matrix in the formula. For general non-stationary flow, the local equilibrium distribution (2.8a), and therefore  $\mathcal{P}$ , are time-dependent. Then, instead of [6] (2.11), for the decomposition of  $e^{\mathcal{L}t}$  the formula [5] (2.4.1) is obtained:

$$e^{\mathcal{L}t} = e^{\mathcal{L}t} \mathcal{P}(t) + \int_0^t dt' e^{\mathcal{L}t'} \mathcal{P}(t') (\mathcal{L} - \dot{\mathcal{P}}(t')) (1 - \mathcal{P}(t')) \mathcal{G}(t', t) + (1 - \mathcal{P}(t)) \mathcal{G}(0, t) \quad (2.11)$$

$$\mathcal{G}(t', t) = \exp_- \left( \int_{t'}^t dt'' \mathcal{L} (1 - \mathcal{P}(t'')) \right) \quad (2.12)$$

$\mathcal{G}(t', t)$  is a time-ordered exponential Operator which describes the time dependence of the dissipative part of the equation of motion. The analysis in [5] consists in applying (2.4) to  $a$ , using (2.11). By averaging over the initial probability density, and after some manipulations, Grabert's generalized transport equation [5], (2.5.17) is obtained. It is postulated in POT that the initial probability density is of the form of the 'relevant probability density', which for simple fluids is defined to be the local equilibrium density (2.8a). Grabert states that this should not be considered a general restriction of the method but a means to form the general particle system into the type specially considered (the simple fluid here); see [5], sec. 2.2. It is shown that in this case the last term in (2.11) vanishes after averaging. - Finally

the part of the formula pertaining to the momentum density is taken, and (1.2) is used. The result, [5] (8.1.12), (8.1.13), reads with the notations employed here:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla P + \nabla \cdot \mathbf{R} \quad (2.13)$$

$$\mathbf{R}(\mathbf{x}, t) = \int_0^t dt' \int d\mathbf{x}' \mathbf{S}(\mathbf{x}, \mathbf{x}', t, t') \nabla' \mathbf{u}(\mathbf{x}', t') \quad (2.14)$$

$$\mathbf{S}(\mathbf{x}, \mathbf{x}', t, t') = \beta \langle [\mathcal{G}(t', t)(1 - \mathcal{P}(t))\mathbf{s}(\mathbf{x})](1 - \mathcal{P}(t'))\mathbf{s}(\mathbf{x}') \rangle_{L, t'} \quad (2.15)$$

The microscopic Definition of the pressure  $P(x, t)$  [5] (8.4.7) will not be repeated here.  $\langle \rangle_{L, t}$  denotes an expectation with respect to the local equilibrium distribution at time  $t$ . Equation (2.13) has the formal structure of the hydrodynamical equation; but at the present state of the analysis, it is still an exact equation. It is the approximations performed later on in the formula for the stress tensor  $\mathbf{R}$  which will transform it into an irreversible equation. - In Addition to (2.13), the equation for the particle number density component of  $\langle a \rangle$  yields the continuity equation:

$$\nabla \cdot \mathbf{u} = 0 \quad (2.16)$$

The kernel function  $\mathbf{S}$  of the stress Tensor  $\mathbf{R}$  is a time correlation function in local equilibrium which is a nonlinear functional of  $\mathbf{u}$ . As far as the author knows, presently it is possible to calculate correlation functions for total equilibrium only. Therefore, it has been necessary to expand  $\mathbf{S}$  into a functional power series in  $\mathbf{u}$ . As has been explained in section 3 of [6], the expansion can be done with the set  $b(t)$  of conjugate parameters of the local equilibrium formula (2.8b). The expansion is performed at the point  $b = b_0$ , which corresponds to  $\mathbf{u} = 0$ :

$$b_0 = \left\{ -\beta \frac{\mu}{m}, \beta, 0 \right\} \quad (2.17)$$

$$b - b_0 = \left\{ \beta \frac{1}{2} u^2, 0, -\beta \mathbf{u} \right\} \quad (2.18)$$

The power expansion of  $S$  reads:

$$\begin{aligned} \mathbf{S} &= \mathbf{S}|_{b_0} + \frac{\delta \mathbf{S}}{\delta b}|_{b_0} * (b - b_0) + \frac{1}{2!} \frac{\delta^2 \mathbf{S}}{\delta b \delta b}|_{b_0} * * \{(b - b_0), (b - b_0)\} + \dots \\ &= \mathbf{S}^{(0)} + \mathbf{S}^{(1)} + \mathbf{S}^{(2)} + \dots \end{aligned} \quad (2.19)$$

The designation of terms in the second row is for later reference. The  $*$  indicates multiplication, summation over five elements and integration over space and time. For the present purpose, the expansion is cut after  $\mathbf{S}^{(2)}$ . When these terms are inserted into (2.14), one obtains corresponding parts  $\mathbf{R}^{(1)}$ ,  $\mathbf{R}^{(2)}$ ,  $\mathbf{R}^{(3)}$  of  $\mathbf{R}$ ; the index again describing the order in  $\mathbf{u}$ . The elements of  $b_0$  are the conjugate parameters of the total equilibrium ensemble. Thus, when  $S$  and its derivatives are taken at  $b = b_0$ , the quantities in the integrand resemble total equilibrium space-time correlation functions. In [6], these quantities have been calculated by Kawasaki technique [7]; though this theory still contains certain intuitive elements. In the present state of the project, for correlation functions the multilinear mode-coupling theory (MCT) of Schofield and co-workers has been used [8]. For the 3-point correlations in  $\mathbf{S}^{(1)}$ , the results of the two methodes coincide; for 4-point quantities, there are differences. - In order to calculate  $\mathbf{S}^{(1)}$ , it is sufficient to restrict  $b - b_0$  to the last element of (2.18), since the first element is of second order in  $\mathbf{u}$ ; but for  $\mathbf{S}^{(2)}$ , this element of  $\mathbf{S}^{(1)}$  has to be added.

It is an important test for the calculation method that (2.13), correctly approximated, should yield the Navier-Stokes equation. The linear part  $\mathbf{R}^{(1)}$  of the stress tensor has been calculated by several authors including Grabert [5], sec. 4; the Stokes form of the stress tensor is obtained, with a microscopic definition of the friction matrix. There remains another detail of the argumentation: The left-hand side of equation (2.13) is of second order in the velocity. Therefore, in order for the equation accurately derived, it must be shown that the second-order part  $\mathbf{R}^{(2)}$  of the stress tensor vanishes. The present author published several papers in ArXiv [9] dealing with this topic, and finally could show that  $\mathbf{R}^{(2)} = 0$ , so that the Navier-Stokes equation derives correctly.

### 3 Third-order term of the stress tensor

Derivation of the third-order term provides considerable effort; a computer algebra system (Mathematica) has been used for most of the calculations. In order to describe the third-order term, we switch from the stress tensor  $\mathbf{R}$  to the friction force  $\mathbf{D} = \nabla \cdot \mathbf{R}$ . By introducing the corresponding term of (2.19) into (2.14), we obtain a formula for the third-order term  $\mathbf{D}_3$  of  $\mathbf{D}$ . Again, only the  $\mathbf{u}$ -part of (2.18) has to be considered. For the detailed investigation, we need the indexed form of the formula:

$$(D_3)_a(\mathbf{x}, t) = \frac{\beta^2}{2} \nabla_c \int_0^t dt' \int d\mathbf{x}' \int_0^\infty dt'' \int d\mathbf{x}'' \int_0^\infty dt''' \int d\mathbf{x}''' \frac{\delta^2 S_{abcd}(\mathbf{x}, \mathbf{x}', t, t')}{\delta b_e(\mathbf{x}'', t'') \delta b_f(\mathbf{x}''', t''')} \Big|_{b_0} \times \\ \times \nabla'_d \mathbf{u}_b(\mathbf{x}', t') \mathbf{u}_e(\mathbf{x}'', t'') \mathbf{u}_f(\mathbf{x}''', t''') \quad (3.1)$$

Latin letters denote indices which run over 3 elements; by contrast, greek indices run from 1 to 5. - In order to continue, the second-order functional derivative of  $\mathbf{S}$  is to be calculated. By [9] (second paper), the first-order derivative consists of two parts. The parametric part is given by (A7), already specialized to  $\mathbf{u} = 0$ ; here we need the general form:

$$\left[ \frac{\delta S_{abcd}}{\delta b_\epsilon(\mathbf{x}'', t'')} \right]_p = -\delta(t'' - t') \beta \langle [\mathcal{G}(t', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') \delta a_\epsilon(\mathbf{x}'', t') \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \quad (3.2)$$

$$\mathcal{Q}(t) = 1 - \mathcal{P}(t) \quad (3.3)$$

$$\delta a_\epsilon(\mathbf{x}, t) = a_\epsilon(\mathbf{x}) - \langle a_\epsilon(\mathbf{x}) \rangle_{L, t} \quad (3.4)$$

The functional part is [9] (second paper) (A14):

$$\left[ \frac{\delta S_{abcd}}{\delta b_\epsilon(\mathbf{x}'', t'')} \right]_f = \beta \Theta(t'' - t') \Theta(t - t'') \langle [\mathcal{G}(t', t'') \mathcal{L} \mathcal{P}(t'') \delta a_\epsilon(\mathbf{x}'', t'') \mathcal{Q}(t'') \mathcal{G}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \quad (3.5)$$

For later reference, the first-order derivatives are formulated with respect to all five conjugated parameters (the greek index  $\epsilon$  is used). - These formulas, as well as the starting formula (2.15), contain operator chains which are, as usual, written like products. The question arises whether I can, when differentiating a chain, use the product rule; in other words, if  $\mathcal{A}$  and  $\mathcal{B}$  are operators, whether it is correct to write:

$$\frac{\delta \mathcal{A} \mathcal{B}}{\delta b} = \frac{\delta \mathcal{A}}{\delta b} \mathcal{B} + \mathcal{A} \frac{\delta \mathcal{B}}{\delta b} \quad (3.6)$$

Since I could not find a general proof in the literature, I checked (3.6) for any two consecutive operators appearing in (3.2), (3.5), and found it correct in all cases. The calculations are partly long and tedious and will not be shown here. - From (3.2), (3.5), the second-order functional derivatives are to be calculated. The formulas depend on  $b$  by 6 and 8 'factors', respectively; including, in each formula, the dependence of the local equilibrium probability density. The corresponding terms are given in Appendix A, (A.2) to (A.15). In the course of the calculation, for  $\mathcal{G}(t', t)$  an identity formula is needed which can be checked by discretizing it:

$$\mathcal{Q}(t') \mathcal{G}(t', t) = \mathcal{Q}(t') \mathcal{G}(t', t) \mathcal{Q}(t) \quad (3.7)$$

Three of the terms are found to vanish; thus, the second-order derivative of  $S$ , and therefore the third-order term of  $R$  consist of 11 terms. Finally to be added are parts of the first-order derivative (3.2), (3.5) which stem from  $\epsilon = 1$ . - For application in (3.1), all terms have to be taken at  $b = b_0$ . This has effects listed below:

- expectations in local equilibrium change into those in total equilibrium, denoted by  $\langle \rangle_0$ ;
- projection operators  $\mathcal{P}(t)$ ,  $\mathcal{Q}(t)$  change to time-independent operators  $\mathcal{P}$ ,  $\mathcal{Q}$ ;
- $\delta a_\epsilon(x, t)$  changes to  $a_\epsilon(x)$ ;
- $\mathcal{G}(t', t'')$  changes to the exponential operator  $\tilde{g}(t)$  with  $t = t'' - t'$ :

$$\tilde{g}(t) = e^{\mathcal{L} \mathcal{Q} t} \quad (3.8)$$

This operator has to be substituted by the similar operator  $g(t)$ :

$$g(t) = e^{\mathcal{Q}\mathcal{L}t} \quad (3.9)$$

The reason is that for  $g(t)$  there is an identity which substitutes it further to the non-projected exponential operator. - The substitution succeeds for all 11 terms with the aid of one of the following identities:

$$\mathcal{Q}\tilde{g}(t) = g(t)\mathcal{Q} \quad (3.10a)$$

$$\tilde{g}(t)\mathcal{L} = \mathcal{L}g(t) \quad (3.10b)$$

$$\tilde{g}(t)\mathcal{Q}\mathcal{L} = -\mathcal{P}\mathcal{L} + \mathcal{L}g(t) \quad (3.10c)$$

Next, we change from the microscopic variables  $a, \mathbf{s}$  to the corresponding ortho-normalized quantities  $h, \mathbf{r}$ ; this extracts from each term the factor  $(\frac{\rho}{\beta})^2$ . Together with the factor  $\beta$  which appears explicitly in each factor, the pre-factor in (3.1) changes to  $\frac{1}{2}\rho^2\beta$ . The gradient operators appearing in (3.1) are transferred to the kernel function; by partial integration, this alters the sign of the expression. In the formula, we then have the expressions  $\nabla_c r_{ac}(\mathbf{x}), \nabla'_d r_{bd}(\mathbf{x}')$  which we again denote  $r_a(\mathbf{x}), r_b(\mathbf{x}')$  respectively; the distinction to the original  $r$ 's is by the number of indices appended. Finally, we change to Fourier space. The expression now reads:

$$(D_3)_0(t) = -\frac{1}{2}\rho^2\beta \int_0^t dt' \int_0^\infty dt'' \int_0^\infty dt''' K(r_0, r_1^*, h_2^*, h_3^*, t, t', t'', t''') \mathbf{u}_1(t') \mathbf{u}_2(t'') \mathbf{u}_3(t''') \quad (3.11)$$

Here we have introduced number indices. A number index is a combined index which contains an index and a wave number variable. Moreover, double indices contain a summation over the indexed elements and an integration over the corresponding wave number space, together with the factor  $\frac{1}{(2\pi)^3}$ . The kernel function  $K$  consists of the 11 Terms presented in Appendix A, (A.16) to (A.26). For shortness, I used the abbreviation  $\hat{\mathbf{r}} = \mathcal{Q}\mathbf{r}$ .

## 4 Preparation of the kernel function

Starting with this section all expectations are with respect to total equilibrium. We can therefore omit the  $_0$  at the expectation symbol. - The kernel function has to be brought into a form so that it contains time correlation functions which can be evaluated by MCT. These are of the general form  $\langle [f(t)A]B^* \rangle_0$  mit  $f(t) = e^{\mathcal{L}t}$ . As a first step, the projection operators  $\mathcal{P}$  and  $\mathcal{Q}$  are to be substituted. As can be seen in the formulas in the appendix, the terms contain 0 to 4 of these operators (the operator contained in  $\hat{\mathbf{r}}$  remains unaltered). With the reformulations done in the last section, the operators now read:

$$\mathcal{P}A = \langle A \rangle + \langle Ah_4^* \rangle h_4 \quad (4.1)$$

$$\mathcal{Q}A = A - \langle A \rangle - \langle Ah_4^* \rangle h_4 \quad (4.2)$$

Generally, the indices contained in the number indices in these formulas must be greek (running over the full list of 5 elements); it then would follow that I have to insert  $\delta h_4$  instead of  $h_4$ . Starting with the consecutive paper, we will restrict the calculation to shear mode flow (as opposed to sound and heat modes); it will then be seen that the indices in (4.1), (4.2) are latin. - Each substitution involves a new doubled number index. If several substitutions are to be done in the same term, the number indices have to be denoted differently. Therefore, the substitutions were performed in 4 consecutive steps. After each step, bracket products were expanded, and certain auxiliary formulas applied:

$$\begin{aligned} \langle h_1 \hat{r}_2 \rangle &= \langle (\mathcal{Q}h_1)r_2 \rangle = 0 \\ \langle \mathcal{L}A \rangle &= 0 \\ \langle \mathcal{L}g(t)h \rangle &= 0 \\ \langle r \rangle &= -\langle \mathcal{L}h \rangle = 0 \\ \mathcal{L}h_1 &= -\hat{r}_1 - \langle r_1 h_4 \rangle h_4 \end{aligned} \quad (4.3)$$

The third and fourth formula are special cases of the second. - The result for the 11 terms, now denoted  $K_1, \dots, K_{11}$ , is presented in the appendix, (A.27). I continue to have the kernel function divided in this manner, for this makes checkups of the calculation somewhat easier. The  $\delta$ - and  $\theta$ -factors are temporarily omitted; they will be added again when time integration is considered.

It is seen that the time correlations in these formulas belong to one of two groups described by:

$$\langle [g(t)\hat{r}_0]B^* \rangle \quad (4.4a)$$

$$\langle [\mathcal{L}g(t)h_0]B^* \rangle \quad (4.4b)$$

$B$  is a certain microscopic function. The formulas are still built with the operator  $g(t)$  and are to be transformed into those with  $f(t)$ . This will be done in a number of steps. We start with (4.4a). First, we have:

$$\langle [g(t)\hat{r}_0]h_1^* \rangle = 0 \quad (4.5)$$

(3.10a) and  $\mathcal{Q}h_1 = 0$  have been used. Next, we introduce a symbol for the memory function of the process:

$$\Gamma_{01}(t) \doteq \langle [g(t)\hat{r}_0]\hat{r}_1^* \rangle \quad (4.6)$$

The memory function is, as usual, "localized" in time, i. e. approximated by:

$$\Gamma_{01}(t) = \tilde{\gamma}_{01}\delta(t) \quad (4.7)$$

$$\tilde{\gamma}_{01} = (2\pi)^3\delta(\mathbf{k}_0 - \mathbf{k}_1)\gamma_{01} \quad (4.8)$$

$\gamma_{01}$ , when inserted into (3.11), for incompressible fluid reduces to (written with indices  $a, b$ ):

$$\gamma_{01} = k_0^2\delta_{ab} \quad (4.9)$$

$\delta_{ab}$  is the Kronecker symbol. - Next, we need an operator identity described in [10], Appendix B2; in the time-dependent form, and applied to the operators defined in this paper, it reads:

$$g(t) = f(t) - \int_0^t dt' f(t')\mathcal{P}\mathcal{L}g(t-t') \quad (4.10)$$

We obtain the correlation function identity, for any microscopic function  $B$ :

$$\langle [g(t)\hat{r}_0]B^* \rangle = \langle [f(t)\hat{r}_0]B^* \rangle - \int_0^t dt' \Gamma_{04}(t-t')\langle [f(t')h_4]B^* \rangle \quad (4.11)$$

After localization of the memory function, we get:

$$\langle [g(t)\hat{r}_0]B^* \rangle = \langle [f(t)\hat{r}_0]B^* \rangle - \tilde{\gamma}_{04}\langle [f(t)h_4]B^* \rangle \quad (4.12)$$

The first term on the rhs of (4.11) is still to be transformed. The fifth formula (4.3) together with (2.4) and  $\tilde{\omega}_{04} = \langle r_0 h_4^* \rangle$  yields:

$$\hat{r}_0 = -\partial_t h_0 - \tilde{\omega}_{04} h_4 \quad (4.13)$$

It is necessary to write the constant matrices  $\tilde{\omega}, \tilde{\kappa}$  with a tilde since they show a form similar to (4.8). - We obtain:

$$\langle [g(t)\hat{r}_0]B^* \rangle = -\partial_t \langle [f(t)h_0]B^* \rangle - \tilde{\kappa}_{04}\langle [f(t)h_4]B^* \rangle \quad (4.14)$$

$$\tilde{\kappa}_{04} = \tilde{\omega}_{04} + \tilde{\gamma}_{04} \quad (4.15)$$

We denote with  $C_n$  die time correlation functions which are to be calculated by MCT:

$$(C_n)_{01\dots n-1}(t) \doteq \langle [f(t)h_0]h_1^* \dots h_{n-1}^* \rangle \quad (4.16)$$

Then, we have the following applications of (4.14):

$$\langle [g(t)\hat{r}_0]h_1^* h_2^* \rangle = -\partial_t (C_3)_{012}(t) - \tilde{\kappa}_{08}(C_3)_{812}(t) \quad (4.17)$$

$$\langle [g(t)\hat{r}_0]h_1^* h_2^* h_3^* \rangle = -\partial_t (C_4)_{0123}(t) - \tilde{\kappa}_{08}(C_4)_{8123}(t) \quad (4.18)$$

Somewhat more complicated is the transformation if  $B$  contains  $\hat{r}$  as a factor. Again, (4.13) is applied, together with  $\tilde{\omega}_{04}^* = -\tilde{\omega}_{04}$ :

$$\begin{aligned}
\langle [g(t)\hat{r}_0]\hat{r}_1^*h_2^* \rangle &= -\partial_t \langle [f(t)h_0]\hat{r}_1^*h_2^* \rangle - \tilde{\kappa}_{04} \langle [f(t)h_4]\hat{r}_1^*h_2^* \rangle \\
&= \partial_t \langle [f(t)h_0](\partial_t h_1^*)h_2^* \rangle - \tilde{\omega}_{17} \partial_t \langle [f(t)h_0]h_7^*h_2^* \rangle + \tilde{\kappa}_{04} (\langle [f(t)h_4](\partial_t h_1^*)h_2^* \rangle - \tilde{\omega}_{17} \langle [f(t)h_4]h_7^*h_2^* \rangle)
\end{aligned} \quad (4.19)$$

The first and the third term need further transformation. Formula (3.11) shows certain symmetries concerning time and wave number integration. Thus, the following formula is correct if it is only used within (3.11):

$$(\partial_t h_1^*)h_2^* = \frac{1}{2}((\partial_t h_1^*)h_2^* + h_1^*\partial_t h_2^*) = \frac{1}{2}\partial_t(h_1^*h_2^*) \quad (4.20)$$

We obtain, for the correlation function (4.19):

$$\begin{aligned}
\langle [g(t)\hat{r}_0]\hat{r}_1^*h_2^* \rangle &= \\
&= -\frac{1}{2}\partial_{tt}(C_3)_{012}(t) - \tilde{\omega}_{17}\partial_t(C_3)_{072}(t) - \tilde{\kappa}_{04}(\frac{1}{2}\partial_t(C_3)_{412}(t) + \tilde{\omega}_{17}(C_3)_{472}(t))
\end{aligned} \quad (4.21)$$

We now switch to time correlations belonging to group (4.4b). We need an operator identity similar to (4.10) which again can be found in [10], Appendix 2B. Applied to this paper, it reads:

$$g(t) = f(t) - \int_0^t dt' g(t') \mathcal{P} \mathcal{L} f(t-t') \quad (4.22)$$

We obtain a formula similar to (4.11):

$$\langle [\mathcal{L}g(t)h_0]B^* \rangle = \langle [\mathcal{L}f(t)h_0]B^* \rangle - \int_0^t dt' \langle [\mathcal{L}f(t-t')h_0]h_4^* \rangle \langle [\mathcal{L}g(t')h_4]B^* \rangle \quad (4.23)$$

For shortness, we introduce some new denotations:

$$\Psi_0(t) = \langle [\mathcal{L}g(t)h_0]B^* \rangle \quad (4.24)$$

$$\Phi_0(t) = \langle [\mathcal{L}f(t)h_0]B^* \rangle = \partial_t \langle [f(t)h_0]B^* \rangle \quad (4.25)$$

Then, applying (4.13), (4.23) reads:

$$\Psi_0(t) = \Phi_0(t) - \int_0^t dt' \partial_t (C_2)_{04}(t-t') \Psi_4(t') \quad (4.26)$$

This is an integral equation for  $\Psi$  which can be formally solved by temporarily switching to Laplace space. The result is:

$$\Psi_0(t) = \Phi_0(t) + \tilde{\kappa}_{04} \int_0^t dt' \Phi_4(t') \quad (4.27)$$

Taking (4.25) into account, we can integrate (4.23):

$$\langle [\mathcal{L}g(t)h_0]B^* \rangle = \partial_t \langle [f(t)h_0]B^* \rangle + \tilde{\kappa}_{04} (\langle [f(t)h_4]B^* \rangle - \langle h_4 B^* \rangle) \quad (4.28)$$

This is the basic formula for time correlations of form (4.4b). Taking  $B = h_1$ , we have:

$$\langle [\mathcal{L}g(t)h_0]h_1^* \rangle = \partial_t (C_2)_{01}(t) + \tilde{\kappa}_{07} ((C_2)_{71}(t) - \delta_{71}) \quad (4.29)$$

The lowest-order non-projected time correlation function  $C_2$  obeys a simple differential equation ([9], third paper, (1.11)):

$$\partial_t (C_2)_{01}(t) = -\tilde{\kappa}_{07} (C_2)_{71}(t) \quad (4.30)$$

This is introduced into (4.29):

$$\langle [\mathcal{L}g(t)h_0]h_1^* \rangle = -\tilde{\kappa}_{01} \quad (4.31)$$

The lowest-order time correlation of type (4.4b) is a constant. - For the next-higher order, we find from (4.28):

$$\langle [\mathcal{L}g(t)h_0]h_1^*h_2^* \rangle = \partial_t(C_3)_{012}(t) + \tilde{\kappa}_{07}((C_3)_{712}(t) - \tilde{j}_{712}) \quad (4.32)$$

$$\tilde{j}_{012} = (C_3)_{012}(t=0) \quad (4.33)$$

Again the situation is somewhat more complicated if  $B$  contains the projected flux  $\hat{r}$ . For the lowest order, we obtain, by applying (4.28):

$$\langle [\mathcal{L}g(t)h_0]\hat{r}_1^*h_2^* \rangle = \partial_t\langle [f(t)h_0]\hat{r}_1^*h_2^* \rangle + \tilde{\kappa}_{04}(\langle [f(t)h_4]\hat{r}_1^*h_2^* \rangle - \langle h_4\hat{r}_1^*h_2^* \rangle) \quad (4.34)$$

We apply (4.13):

$$\begin{aligned} \langle [\mathcal{L}g(t)h_0]\hat{r}_1^*h_2^* \rangle &= -\partial_t\langle [f(t)h_0](\partial_t h_1^*)h_2^* \rangle + \tilde{\omega}_{17}\partial_t\langle [f(t)h_0]h_7^*h_2^* \rangle + \\ &\quad + \tilde{\kappa}_{04}(-\langle [f(t)h_4](\partial_t h_1^*)h_2^* \rangle + \tilde{\omega}_{17}\langle [f(t)h_4]h_7^*h_2^* \rangle - \langle h_4\hat{r}_1^*h_2^* \rangle) \end{aligned} \quad (4.35)$$

We introduce a notation for the static correlation appearing in the formula:

$$\tilde{s}_{124} = \langle \hat{r}_1 h_2 h_4^* \rangle = -\langle \hat{r}_1 h_2 h_4^* \rangle^* = -\langle h_4 \hat{r}_1^* h_2^* \rangle \quad (4.36)$$

Finally, we can apply a symmetry consideration as in (4.19); by using (4.20) we obtain:

$$\begin{aligned} \langle [\mathcal{L}g(t)h_0]\hat{r}_1^*h_2^* \rangle &= \frac{1}{2}\partial_{tt}(C_3)_{012}(t) + \tilde{\omega}_{17}\partial_t(C_3)_{072}(t) + \\ &\quad + \tilde{\kappa}_{04}(\frac{1}{2}\partial_t(C_3)_{412}(t) + \tilde{\omega}_{17}(C_3)_{472}(t) + \tilde{s}_{124}) \end{aligned} \quad (4.37)$$

For the next-higher order correlation we have, by (4.28):

$$\langle [\mathcal{L}g(t)h_0]\hat{r}_1^*h_2^*h_3^* \rangle = \partial_t\langle [f(t)h_0]\hat{r}_1^*h_2^*h_3^* \rangle + \tilde{\kappa}_{04}(\langle [f(t)h_4]\hat{r}_1^*h_2^*h_3^* \rangle - \langle h_4\hat{r}_1^*h_2^*h_3^* \rangle) \quad (4.38)$$

Introduction of (4.13) yields:

$$\begin{aligned} \langle [\mathcal{L}g(t)h_0]\hat{r}_1^*h_2^*h_3^* \rangle &= -\partial_t\langle [f(t)h_0](\partial_t h_1^*)h_2^*h_3^* \rangle + \tilde{\omega}_{17}\partial_t\langle [f(t)h_0]h_7^*h_2^*h_3^* \rangle \\ &\quad + \tilde{\kappa}_{04}(-\langle [f(t)h_4](\partial_t h_1^*)h_2^*h_3^* \rangle + \tilde{\omega}_{17}\langle [f(t)h_4]h_7^*h_2^*h_3^* \rangle - \langle h_4\hat{r}_1^*h_2^*h_3^* \rangle) \end{aligned} \quad (4.39)$$

The symmetry consideration which is correct if it is applied within the integral formula (3.11) gives:

$$(\partial_t h_1^*)h_2^*h_3^* = \frac{1}{3}((\partial_t h_1^*)h_2^*h_3^* + h_1^*(\partial_t h_2^*)h_3^* + h_1^*h_2^*(\partial_t h_3^*)) = \frac{1}{3}\partial_t(h_1^*h_2^*h_3^*) \quad (4.40)$$

Again a denotation for the static correlation is introduced:

$$\tilde{s}_{41234} = \langle \hat{r}_1 h_2 h_3 h_4^* \rangle = -\langle \hat{r}_1 h_2 h_3 h_4^* \rangle^* = -\langle h_4 \hat{r}_1^* h_2^* h_3^* \rangle \quad (4.41)$$

The final form for (4.38) is:

$$\begin{aligned} \langle [\mathcal{L}g(t)h_0]\hat{r}_1^*h_2^*h_3^* \rangle &= \frac{1}{3}\partial_{tt}(C_4)_{0123}(t) + \tilde{\omega}_{17}\partial_t(C_4)_{0723}(t) + \\ &\quad + \tilde{\kappa}_{04}(\frac{1}{3}\partial_t(C_4)_{4123}(t) + \tilde{\omega}_{17}(C_4)_{4723}(t) + \tilde{s}_{41234}) \end{aligned} \quad (4.42)$$

The formulas obtained in this section are introduced into the terms (A.27). The Dirac and Heaviside functions omitted are added again. The kernel function  $K$  in (3.11) now consists of 5 main parts:

$$\begin{aligned} K &= \delta(t''' - t')\delta(t'' - t')M_1 \\ &\quad + \delta(t'' - t')\Theta(t''' - t')\Theta(t - t''')M_2 \\ &\quad + \delta(t''' - t'')\Theta(t'' - t')\Theta(t - t'')\delta(t'' - t')M_3 \\ &\quad + \Theta(t''' - t'')\Theta(t - t''')\Theta(t'' - t')\Theta(t - t'')\delta(t'' - t')M_4 \\ &\quad + \Theta(t''' - t')\Theta(t'' - t''')\Theta(t'' - t')\Theta(t - t'')\delta(t''' - t')M_5 \end{aligned} \quad (4.43)$$

The coefficients  $M_1, \dots, M_5$  are presented in the appendix, (A.28). When this is inserted into (3.11), some of the time integrals can be executed. Moreover, we can apply an approximation since the time

scale of the  $\mathbf{u}$ -factors is much larger than that of the equilibrium time correlations. The upper limits of integrals over correlation functions can then be extended to infinity. The formula now reads:

$$(D_3)_0(t) = -\frac{1}{2}\rho^2\beta\mathbf{u}_1(t)\mathbf{u}_2(t)\mathbf{u}_3(t) \times \\ \times \left( \int_0^\infty dt' \Phi_{10123}(t') + \lim_{t \rightarrow \infty} \int_0^t dt' \int_0^{t-t'} dt'' \Phi_{20123}(t', t'') \right) \quad (4.44)$$

with  $\Phi 1 = M_1 + M_3$  and  $\Phi 2 = M_2 + M_4 + M_5$ . The detailed expression are given in the appendix, (A.29). - This is the final form of the part of the third-order term which stems from the term  $\mathbf{S}^{(2)}$  of the kernel function in the expression (2.14) for the stress tensor. As has been mentioned earlier, still to be added is the part of the second-order term which derives from (3.3), (3.6) for  $\epsilon = 1$ .

## 5 Relevant part of the second-order term

When one treats the second-order term in the same way which led to (3.11), one obtains:

$$(D_2)_0(t) = -\frac{1}{2}\rho^{\frac{3}{2}}\beta^{\frac{1}{2}} \left( \frac{\partial p}{\partial \rho} \right)^{-\frac{1}{2}} \int_0^t dt' \int_0^\infty dt'' (K_2)_{012}(t, t', t'') u_1(t') u_q(t'') \quad (5.1)$$

In this section, the number index 2 contains a component index which has the value 1, if not stated otherwise. The pre-factor contains a part which results from switching to normalized variables in the correlation functions.  $u_q$  is the Fourier transform of  $u^2$  for the wave number variable  $\mathbf{k}''$ . The kernel function  $K_2$  is the sum of (3.2) and (3.5), taken at  $b = b_0$  for  $\epsilon = 1$ ; (3.10a) has been used to substitute the operator  $\tilde{g}$ :

$$(K_2)_{012}(t, t', t'') = -\delta(t'' - t') \langle [g(t - t') \hat{r}_0] \hat{r}_1^* \delta h_2^* \rangle + \\ + \Theta(t'' - t') \Theta(t - t'') \langle [g(t'' - t') \mathcal{QLP} \delta h_2^* g(t - t'') \hat{r}_0] \hat{r}_1^* \rangle \quad (5.2)$$

For the first term, we have:

$$\langle [g(t - t') \hat{r}_0] \hat{r}_1^* \delta h_2^* \rangle = \langle [g(t - t') \hat{r}_0] \hat{r}_1^* h_2^* \rangle - \langle [g(t - t') \hat{r}_0] \hat{r}_1^* \rangle \langle h_2^* \rangle \quad (5.3)$$

The operator  $\mathcal{P}$  in the second term in (5.2) is evaluated:

$$\langle [g(t'' - t') \mathcal{QLP} \delta h_2^* g(t - t'') \hat{r}_0] \hat{r}_1^* \rangle \\ = -\langle [g(t - t'') \hat{r}_0] h_2^* h_3^* \rangle \langle [g(t'' - t') \hat{r}_3] \hat{r}_1^* \rangle \quad (5.4)$$

The identity  $\mathcal{QL} \delta h_3 = -\mathcal{Q}r_3 = -\hat{r}_3$  has been used. Originally, the full form (4.1) with  $\delta h$  instead of  $h$  has been used in the first factor; but it is found that the terms with the constant parts  $\langle h_2 \rangle, \langle h_3 \rangle$  vanish, so that the final form is (5.4). - For further preparation the formulas (4.6) and (4.7), (4.17), (4.21) are used, integration over  $t''$  and an approximation because of different time scales is performed:

$$(D_2)_0(t) = -\frac{1}{2}\rho^{\frac{3}{2}}\beta^{\frac{1}{2}} \left( \frac{\partial p}{\partial \rho} \right)^{-\frac{1}{2}} u_1(t) u_q(t) \int_0^\infty dt' (K_2)_{012}(t') \quad (5.5)$$

$K_2$ , now a function of  $t'$  only, is given in the appendix, (A.30).

## 6 Result

The quantity (5.5) has to be added to (4.44) in order to obtain the part of the friction force which is third order in the velocity  $\mathbf{u}$ . We denote it again  $D_3$ :

$$\begin{aligned}
(D_3)_0(t) = & -\frac{1}{2}\rho^2\beta(\mathbf{u}_1(t)\mathbf{u}_2(t)\mathbf{u}_3(t) \times \\
& \times (\int_0^\infty dt' \Phi_{0123}(t') + \lim_{t \rightarrow \infty} \int_0^t dt' \int_0^{t-t'} dt'' \Phi_{0123}(t', t'')) \\
& + (\rho\beta\partial_\rho P|_\beta)^{-\frac{1}{2}} u_1(t)uq(t) \int_0^\infty dt' (K_2)_{012}(t'))
\end{aligned} \tag{6.1}$$

This formula is the final result of this paper. The next step is the determination of the equilibrium correlation functions  $C_3$  and  $C_4$ . The MCT has been applied for this purpose. A formula for  $C_3$  is explicitly given in [8], while  $C_4$  has been calculated with the aid of the rules given in these papers. This will be described in a consecutive paper. - The transformation back to geometric space represented another major effort which resulted in closed formulas for the kernel functions. The formula obtained contains multiple space integrals; therefore, the hydrodynamic equation (2.13) in this approximation is an integro-differential equation which is second-order in the local and third-order in the non-local part. This part of the investigation will be discussed in a another paper.

## A Appendix: Calculation of second order derivative

In this appendix, formulas are presented which appear in the course of the analysis in the main part of the paper, but are too voluminous to be given there. - First, the terms of the second order functional derivative of the stress tensor kernel function  $S$  are written, as calculated from (3.2), (3.5). The detailed formula for the second derivative reads:

$$\frac{\delta^2 S_{abcd}(\mathbf{x}', t')}{\delta b_e(\mathbf{x}'', t'')\delta b_f(\mathbf{x}''', t''')} \tag{A.1}$$

For shortness, on the left hand side, indices and variables are omitted. Second order derivatives with respect to the momentum part of the conjugated parameters are needed only (that is, the greek index  $\epsilon$  is switched to  $e$ ). For each term, the first step is just the definition of the quantity.

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,1)} &= -\delta(t'' - t')\beta \text{tr} \left\{ \frac{\delta f_L(t')}{\delta b_f(\mathbf{x}''', t''')} [\tilde{\mathcal{G}}(t', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t') \right\} \\
&= \delta(t''' - t')\delta(t'' - t')\beta \langle [\tilde{\mathcal{G}}(t', t) \hat{s}_{ac}(\mathbf{x}, t)] [\mathcal{Q}(t') [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t')] \delta a_f(\mathbf{x}''', t') \rangle_{L, t'}
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,2)} &= -\delta(t'' - t')\beta \langle [\tilde{\mathcal{G}}(t', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \frac{\delta(\delta a_e(\mathbf{x}'', t'))}{\delta b_f(\mathbf{x}''', t''')} \rangle_{L, t'} \\
&= -\delta(t''' - t')\delta(t'' - t')\beta \langle [\tilde{\mathcal{G}}(t', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \langle a_e(\mathbf{x}'') \delta a_f(\mathbf{x}''', t') \rangle_{L, t'}
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,3)} &= -\delta(t'' - t')\beta \langle [\tilde{\mathcal{G}}(t', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') \left[ \frac{\delta \mathcal{Q}(t')}{\delta b_f(\mathbf{x}''', t''')} s_{bd}(\mathbf{x}') \right] \delta a_e(\mathbf{x}'', t') \rangle_{L, t'} \\
&= -\delta(t''' - t')\delta(t'' - t') \times \\
&\quad \times \beta \langle [\tilde{\mathcal{G}}(t', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') [\mathcal{P}(t') \delta a_f(\mathbf{x}''', t') \mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t') \rangle_{L, t'}
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,4)} &= -\delta(t'' - t')\beta \langle [\tilde{\mathcal{G}}(t', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \frac{\delta \mathcal{Q}(t')}{\delta b_f(\mathbf{x}''', t''')} [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t') \rangle_{L, t'} \\
&- \delta(t''' - t')\delta(t'' - t') \times \\
&\quad \times \beta \langle [\tilde{\mathcal{G}}(t', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{P}(t') \delta a_f(\mathbf{x}''', t') \mathcal{Q}(t') [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t') \rangle_{L, t'}
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,5)} &= -\delta(t'' - t') \beta \langle [\tilde{\mathcal{G}}(t', t) \frac{\delta \mathcal{Q}(t)}{\delta b_f(\mathbf{x}''', t''')} s_{ac}(\mathbf{x})] \mathcal{Q}(t') [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t') \rangle_{L, t'} \\
&= -\delta(t''' - t) \delta(t'' - t') \times \\
&\quad \beta \langle [\tilde{\mathcal{G}}(t', t) \mathcal{P}(t) \delta a_f(\mathbf{x}'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t') \rangle_{L, t'} \\
&= 0
\end{aligned} \tag{A.6}$$

The last step is correct because of (3.7) and  $\mathcal{Q}\mathcal{P} = 0$ .

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(p,6)} &= -\delta(t'' - t') \beta \langle [\frac{\delta \tilde{\mathcal{G}}(t', t)}{\delta b_f(\mathbf{x}''', t''')} \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t') \rangle_{L, t'} \\
&= -\delta(t'' - t') \Theta(t''' - t') \Theta(t - t''') \times \\
&\quad \times \beta \langle [\tilde{\mathcal{G}}(t', t''') \mathcal{L}\mathcal{P}(t''') \delta a_f(\mathbf{x}'', t''') \mathcal{Q}(t''') \tilde{\mathcal{G}}(t''', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_e(\mathbf{x}'', t') \rangle_{L, t'}
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,1)} &= \Theta(t'' - t') \Theta(t - t'') \times \\
&\quad \times \beta \text{tr} \left\{ \frac{\delta f_L(t')}{\delta b_f(\mathbf{x}''', t''')} [\tilde{\mathcal{G}}(t', t'') \mathcal{L}\mathcal{P}(t'') \delta a_e(\mathbf{x}'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \right\} \\
&= -\delta(t''' - t') \Theta(t'' - t') \Theta(t - t'') \times \\
&\quad \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L}\mathcal{P}(t'') \delta a_e(\mathbf{x}'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] [\mathcal{Q}(t') s_{bd}(\mathbf{x}')] \delta a_f(\mathbf{x}''', t') \rangle_{L, t'}
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,2)} &= \Theta(t'' - t') \Theta(t - t'') \times \\
&\quad \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L}\mathcal{P}(t'') \delta a_e(\mathbf{x}'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \frac{\delta \mathcal{Q}(t')}{\delta b_f(\mathbf{x}''', t''')} s_{bd}(\mathbf{x}') \rangle_{L, t'} \\
&= \delta(t''' - t') \Theta(t'' - t') \Theta(t - t'') \times \\
&\quad \times \beta \langle \tilde{\mathcal{G}}[t', t'') \mathcal{L}\mathcal{P}(t'') \delta a_e(\mathbf{x}'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{P}(t') \delta a_f(\mathbf{x}''', t') \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'}
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,3)} &= \Theta(t'' - t') \Theta(t - t'') \times \\
&\quad \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L}\mathcal{P}(t'') \delta a_e(\mathbf{x}'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \frac{\delta \mathcal{Q}(t)}{\delta b_f(\mathbf{x}''', t''')} s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \\
&= \delta(t''' - t) \Theta(t'' - t') \Theta(t - t'') \times \\
&\quad \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L}\mathcal{P}(t'') \delta a_e(\mathbf{x}'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{P}(t) \delta a_f(\mathbf{x}''', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \\
&= 0
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
\left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,4)} &= \Theta(t'' - t') \Theta(t - t'') \times \\
&\quad \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L}\mathcal{P}(t'') \delta a_e(\mathbf{x}'', t'') \frac{\delta \mathcal{Q}(t'')}{\delta b_f(\mathbf{x}''', t''')} \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \\
&\quad \delta(t''' - t'') \Theta(t'' - t') \Theta(t - t'') \times \\
&\quad \times \beta \langle \tilde{\mathcal{G}}[t', t'') \mathcal{L}\mathcal{P}(t'') \delta a_e(\mathbf{x}'', t'') \mathcal{P}(t'') \delta a_f(\mathbf{x}''', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'}
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
& \left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,5)} = \Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L} \mathcal{P}(t'') \frac{\delta \langle a_e(\mathbf{x}'', t'') \rangle}{\delta b_f(\mathbf{x}''', t''')} \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \\
& = -\Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L} \mathcal{P}(t'') \frac{\delta \langle a_e(x'') \rangle_{L, t''}}{\delta b_f(\mathbf{x}''', t''')} \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \\
& = \delta(t''' - t'') \Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L} \mathcal{P}(t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \langle a_e(x'') \delta a_f(x''', t'') \rangle_{L, t''} \\
& = 0
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
& \left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,6)} = \Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L} \frac{\delta \mathcal{P}(t'')}{\delta b_f(\mathbf{x}''', t''')} \delta a_e(x'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) \hat{s}_{ac}(\mathbf{x})] \mathcal{Q}(t') \hat{s}_{bd}(\mathbf{x}') \rangle_{L, t'} \\
& = -\delta(t''' - t'') \Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L} \mathcal{P}(t'') \delta a_f(\mathbf{x}''', t'') \mathcal{Q}(t'') \delta a_e(x'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) \hat{s}_{ac}(\mathbf{x})] \times \\
& \times \mathcal{Q}(t') \hat{s}_{bd}(\mathbf{x}') \rangle_{L, t'}
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
& \left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,7)} = \Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L} \mathcal{P}(t'') \delta a_e(x'', t'') \mathcal{Q}(t'') \frac{\delta \tilde{\mathcal{G}}(t'', t)}{\delta b_f(\mathbf{x}''', t''')} \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \\
& = \Theta(t''' - t'') \Theta(t - t''') \Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\tilde{\mathcal{G}}(t', t'') \mathcal{L} \mathcal{P}(t'') \delta a_e(x'', t'') \mathcal{Q}(t'') \times \\
& \times \tilde{\mathcal{G}}(t'', t''') \mathcal{L} \mathcal{P}(t''') \delta a_f(\mathbf{x}''', t''') \mathcal{Q}(t''') \tilde{\mathcal{G}}(t''', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'}
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
& \left[ \frac{\delta^2 S}{\delta b \delta b} \right]^{(f,8)} = \Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\frac{\delta \tilde{\mathcal{G}}(t', t'')}{\delta b_f(\mathbf{x}''', t''')} \mathcal{L} \mathcal{P}(t'') \delta a_e(x'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'} \\
& = \Theta(t''' - t') \Theta(t'' - t''') \Theta(t'' - t') \Theta(t - t'') \times \\
& \times \beta \langle [\tilde{\mathcal{G}}(t', t''') \mathcal{L} \mathcal{P}(t''') \delta a_f(\mathbf{x}''', t''') \mathcal{Q}(t''') \tilde{\mathcal{G}}(t''', t'') \times \\
& \times \mathcal{L} \mathcal{P}(t'') \delta a_e(x'', t'') \mathcal{Q}(t'') \tilde{\mathcal{G}}(t'', t) \mathcal{Q}(t) s_{ac}(\mathbf{x})] \mathcal{Q}(t') s_{bd}(\mathbf{x}') \rangle_{L, t'}
\end{aligned} \tag{A.15}$$

Results (A.10), (A.12) are found the same way as (A.6). - Below, the non-zero terms are presented in the form they assume after the reformulation in sec. 3:

$$\left[ \frac{\delta^2 R}{\delta b \delta b} \right]_0^{(p,1)} = -\delta(t''' - t') \delta(t'' - t') \langle [\mathcal{P} r_0 + \mathcal{L} g(t - t') h_0] [\mathcal{Q} \hat{w}_1^* h_2^*] h_3^* \rangle_0 \tag{A.16}$$

$$\left[ \frac{\delta^2 R}{\delta b \delta b} \right]_0^{(p,2)} = -\delta(t''' - t') \delta(t'' - t') \langle [g(t - t') \hat{r}_0] \hat{r}_1^* \rangle_0 \langle h_2^* h_3^* \rangle_0 \tag{A.17}$$

$$\left[ \frac{\delta^2 R}{\delta b \delta b} \right]_0^{(p,3)} = -\delta(t''' - t') \delta(t'' - t') \langle [g(t - t') \hat{r}_0] [\mathcal{P} \hat{r}_1^* h_3^*] h_2^* \rangle_0 \tag{A.18}$$

$$\left[ \frac{\delta^2 R}{\delta b \delta b} \right]_0^{(p,4)} = \delta(t''' - t') \delta(t'' - t') \langle [\mathcal{P} r_0 + \mathcal{L} g(t - t') h_0] \mathcal{P} h_3^* \mathcal{Q} \hat{r}_1^* h_2^* \rangle_0 \tag{A.19}$$

$$\left[\frac{\delta^2 R}{\delta b \delta b}\right]_0^{(p,6)} = -\delta(t''' - t')\Theta(t''' - t')\Theta(t - t''')\langle g[(t''' - t')\mathcal{Q}\mathcal{L}\mathcal{P}h_3^*g(t - t''')\hat{r}_0]\hat{r}_1^*h_2^*\rangle_0 \quad (\text{A.20})$$

$$\left[\frac{\delta^2 R}{\delta b \delta b}\right]_0^{(f,1)} = -\delta(t''' - t')\Theta(t'' - t')\Theta(t - t'')\langle [\mathcal{L}g(t'' - t')\mathcal{P}h_2^*g(t - t'')\hat{r}_0]\hat{r}_1^*h_3^*\rangle_0 \quad (\text{A.21})$$

$$\left[\frac{\delta^2 R}{\delta b \delta b}\right]_0^{(f,2)} = \delta(t''' - t')\Theta(t'' - t')\Theta(t - t'')\langle [\mathcal{L}g(t'' - t')\mathcal{P}h_2^*g(t - t'')\hat{r}_0]\mathcal{P}\hat{r}_1^*h_3^*\rangle_0 \quad (\text{A.22})$$

$$\left[\frac{\delta^2 R}{\delta b \delta b}\right]_0^{(f,4)} = \delta(t''' - t'')\Theta(t'' - t')\Theta(t - t'')\langle [g(t'' - t')\mathcal{Q}\mathcal{L}\mathcal{P}h_2^*\mathcal{P}h_3^*g(t - t'')\hat{r}_0]\hat{r}_1^*\rangle_0 \quad (\text{A.23})$$

$$\left[\frac{\delta^2 R}{\delta b \delta b}\right]_0^{(f,6)} = -\delta(t''' - t'')\Theta(t'' - t')\Theta(t - t'')\langle [g(t'' - t')\mathcal{Q}\mathcal{L}\mathcal{P}h_3^*\mathcal{Q}h_2^*g(t - t'')\hat{r}_0]\hat{r}_1^*\rangle_0 \quad (\text{A.24})$$

$$\begin{aligned} \left[\frac{\delta^2 R}{\delta b \delta b}\right]_0^{(f,7)} &= \Theta(t''' - t'')\Theta(t - t''')\Theta(t'' - t')\Theta(t - t'') \times \\ &\times \langle [g(t'' - t')\mathcal{Q}\mathcal{L}\mathcal{P}h_2^*g(t''' - t'')\mathcal{Q}\mathcal{L}\mathcal{P}h_3^*g(t - t''')\hat{r}_0]\hat{r}_1^*\rangle_0 \end{aligned} \quad (\text{A.25})$$

$$\begin{aligned} \left[\frac{\delta^2 R}{\delta b \delta b}\right]_0^{(f,8)} &= \Theta(t''' - t')\Theta(t'' - t''')\Theta(t'' - t')\Theta(t - t'') \times \\ &\times \langle [g(t''' - t')\mathcal{Q}\mathcal{L}\mathcal{P}h_3^*g(t'' - t''')\mathcal{Q}\mathcal{L}\mathcal{P}h_2^*g(t - t'')\hat{r}_0]\hat{r}_1^*\rangle_0 \end{aligned} \quad (\text{A.26})$$

After evaluation of the projection operators in section 4, we obtain for the 11 terms, now called  $K_1, \dots, K_{11}$  (Symbol  $_0$  omitted from  $\langle \rangle$ ):

$$\begin{aligned} (K_1)_0 &= \langle [\mathcal{L}g(t - t_1)h_0]h_3^*h_4^*\langle \hat{r}_1^*h_2^*h_4 \rangle - \langle [\mathcal{L}g(t - t_1)h_0]\hat{r}_1^*h_2^*h_3^* \rangle \\ (K_2)_0 &= -\langle [g(t - t_1)\hat{r}_0]\hat{r}_1^*\langle h_2^*h_3^* \rangle \\ (K_3)_0 &= -\langle [g(t - t_1)\hat{r}_0]h_2^*h_4^*\langle \hat{r}_1^*h_3^*h_4 \rangle \\ (K_4)_0 &= \langle [\mathcal{L}g(t - t_1)h_0]h_4^*\langle \hat{r}_1^*h_2^*h_3^*h_4 \rangle - \langle [\mathcal{L}g(t - t_1)h_0]h_4^*\langle \hat{r}_1^*h_2^*h_5 \rangle \langle h_3^*h_4h_5^* \rangle \\ (K_5)_0 &= \langle [g(t - t_3)\hat{r}_0]h_3^*h_5^*\langle [g(t_3 - t_1)\hat{r}_5]\hat{r}_1^*h_2^* \rangle \\ (K_6)_0 &= -\langle [g(t - t_2)\hat{r}_0]h_2^*h_4^*\langle [\mathcal{L}g(t_2 - t_1)h_4]\hat{r}_1^*h_3^* \rangle \\ (K_7)_0 &= \langle [g(t - t_2)\hat{r}_0]h_2^*h_5^*\langle [\mathcal{L}g(t_2 - t_1)h_5]h_4^*\langle \hat{r}_1^*h_3^*h_4 \rangle \\ (K_8)_0 &= -\langle [g(t - t_2)\hat{r}_0]h_3^*\langle [g(t_2 - t_1)\hat{r}_5]\hat{r}_1^*\langle h_2^*h_5^* \rangle \\ &\quad + \langle [g(t - t_2)\hat{r}_0]h_3^*h_6^*\langle [g(t_2 - t_1)\hat{r}_5]\hat{r}_1^*\langle h_2^*h_5^*h_6 \rangle \\ (K_9)_0 &= -\langle [\mathcal{L}g(t - t_2)h_0]h_2^*\langle [g(t_2 - t_1)\hat{r}_5]\hat{r}_1^*\langle h_3^*h_5^* \rangle \\ &\quad + \langle [g(t - t_2)\hat{r}_0]h_2^*h_3^*h_5^*\langle [g(t_2 - t_1)\hat{r}_5]\hat{r}_1^* \rangle \\ &\quad + \langle [g(t - t_2)\hat{r}_0]h_2^*h_6^*\langle [g(t_2 - t_1)\hat{r}_5]\hat{r}_1^*\langle h_3^*h_5^*h_6 \rangle \\ (K_{10})_0 &= \langle [g(t - t_3)\hat{r}_0]h_3^*h_7^*\langle [g(t_2 - t_1)\hat{r}_5]\hat{r}_1^*\langle [g(t_3 - t_2)\hat{r}_7]h_2^*h_5^* \rangle \\ (K_{11})_0 &= \langle [g(t - t_2)\hat{r}_0]h_2^*h_7^*\langle [g(t_3 - t_1)\hat{r}_5]\hat{r}_1^*\langle [g(t_2 - t_3)\hat{r}_7]h_3^*h_5^* \rangle \end{aligned} \quad (\text{A.27})$$

In sec. 4, these terms are combined into 5 main parts:

$$\begin{aligned}
(M_1)_{0123} &= -c_{23}\tilde{\gamma}_{01}\delta(t-t') \\
&\quad -2\tilde{s}_{124}(\tilde{\kappa}_{05}(C_3)_{534}(t-t') + \partial_t(C_3)_{034}(t-t')) \\
&\quad -\tilde{\omega}_{18}((C_4)_{7823}(t-t')\tilde{\kappa}_{07} + \partial_t(C_4)_{0823}(t-t')) \\
&\quad -\frac{1}{3}(\tilde{\kappa}_{07}\partial_t(C_4)_{7123}(t-t') + \partial_{tt}(C_4)_{0123}(t-t')) \\
(M_2)_{0123} &= ((C_3)_{824}(t-t'')\kappa_{08} + \partial_t(C_3)_{024}(t-t'')) \\
&\quad \times (\tilde{\kappa}_{45}(s_{513} - s_{135}) \\
&\quad + 2\tilde{\omega}_{16}((C_3)_{763}(t''-t')\tilde{\kappa}_{47} + \partial_t(C_3)_{463}(t''-t')) \\
&\quad + \tilde{\kappa}_{47}\partial_t(C_3)_{713}(t''-t') + \frac{1}{2}\partial_{tt}(C_3)_{413}(t''-t')) \\
(M_3)_{0123} &= \tilde{\gamma}_{51}(\tilde{\kappa}_{08}((C_3)_{836}(t-t'')\tilde{j}_{625} + (C_3)_{826}(t-t'')j_{635}) \\
&\quad + \partial_t(C_3)_{026}(t-t'')\tilde{j}_{635} + \partial_t(C_3)_{036}(t-t'')j_{625} \\
&\quad - (C_4)_{8235}(t-t'')\tilde{\kappa}_{08} - \partial_t(C_4)_{0235}(t-t'')) \\
(M_4)_{0123} &= \tilde{\gamma}_{51}(\tilde{\kappa}_{08}(C_3)_{837}(t-t''') + \partial_t(C_3)_{037}(t-t''')) \\
&\quad \times (\tilde{\kappa}_{79}(C_3)_{925}(t'''-t'') + \partial_t(C_3)_{725}(t'''-t'')) \\
(M_5)_{0123} &= \tilde{\gamma}_{51}(\tilde{\kappa}_{08}(C_3)_{827}(t-t'') + \partial_t(C_3)_{027}(t-t'')) \\
&\quad \times (\tilde{\kappa}_{79}(C_3)_{935}(t''-t''') + \partial_t(C_3)_{735}(t''-t'''))
\end{aligned} \tag{A.28}$$

When the  $M$ -quantities are inserted into (3.11), some of the time integrations can be performed; finally, two parts of the kernel function remain:

$$\begin{aligned}
\Phi_{10123}(t') &= -\tilde{c}_{23}\tilde{\gamma}_{01}\delta(t') \\
&\quad -2\tilde{s}_{124}(\tilde{\kappa}_{05}(C_3)_{534}(t') + \partial_t(C_3)_{034}(t')) \\
&\quad + 2\tilde{j}_{625}\tilde{\gamma}_{51}((\kappa_{08}((C_3)_{836}(t') + \partial_t(C_3)_{036}(t')) \\
&\quad - \tilde{\kappa}_{51}((C_4)_{8235}(t')\tilde{\kappa}_{08} + \partial_t(C_4)_{0235}(t')) \\
&\quad - \frac{1}{3}(\tilde{\kappa}_{07}\partial_t(C_4)_{7123}(t') + \partial_{tt}(C_4)_{0123}(t')) \\
\Phi_{20123}(t', t'') &= 2\tilde{\gamma}_{51}(\tilde{\kappa}_{08}(C_3)_{837}(t') + \partial_t(C_3)_{037}(t')) \\
&\quad \times (\tilde{\kappa}_{79}(C_3)_{925}(t'') + \partial_t(C_3)_{725}(t'')) \\
&\quad + ((C_3)_{824}(t')\tilde{\kappa}_{08} + \partial_t(C_3)_{024}(t')) \\
&\quad \times (\tilde{\kappa}_{45}(\tilde{s}_{513} - \tilde{s}_{135}) \\
&\quad + 2\tilde{\omega}_{16}((C_3)_{763}(t'')\tilde{\kappa}_{47} + \partial_t(C_3)_{463}(t'')) \\
&\quad + \tilde{\kappa}_{47}\partial_t(C_3)_{713}(t'') + \frac{1}{2}\partial_{tt}(C_3)_{413}(t''))
\end{aligned} \tag{A.29}$$

In sec. 5, the part of the second-order (in  $b - b_0$ ) term which adds to the third-order velocity term is calculated. The kernel function reads:

$$\begin{aligned}
(K_2)_{012}(t') &= \tilde{\gamma}_{01}\delta(t)\langle h_2 \rangle \\
&\quad + \tilde{\gamma}_{31}(\partial_t(C_3)_{032}(t) + \tilde{\kappa}_{04}(C_3)_{432}(t)) \\
&\quad + \partial_{tt}(C_3)_{012}(t) + i k_c''\partial_t(C_3)_{012}|_{2''=c}(t) + \tilde{\omega}_{14}\partial_t(C_3)_{042}(t) \\
&\quad + \tilde{\kappa}_{03}(\partial_t(C_3)_{312}(t) + i k_c''(C_3)_{312}|_{2''=c}(t) + \tilde{\omega}_{14}(C_3)_{342}(t))
\end{aligned} \tag{A.30}$$

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